Expand the initial electron wavefunction in energy:

and expand the field in terms of frequency (for guided modes):

with . In addition, write the dispersion around the phase matching point as

For spontaneous emission, the matrix element is

If the initial wavefunction is then the joint electron-photon state is

Trace out over the photon state to obtain for the electron in terms of and the dispersion – without neglecting recoil, finite interaction length, higher order dispersion, etc. Explicitly, we have

with . Next,

with . Taking the delta integral yields the exact result

**Approximation**

Next let’s try to gather analytic results/intuition. First let’s take out all slowly-varying quantities outside the integral:

Also, let’s approximate the sinc argument up to quadratic recoil corrections. Explicitly, this means that:

where is the initial electron’s central energy. Now it is convenient to define the “deviation variables”:

in this manner, is centered around the first loss peak () and around the phase-matching frequency. In terms of these variables we have

Next, use to write:

So

Finally, the argument of the sinc becomes (assuming the dispersion is at most quadratic)

with . So the latter simplifies to:

Therefore, to second-order corrections our loss peak is:

if further :

It is instructive to study the behavior of this function. For this let us write

and write the integrand in terms of dimensionless quantities via:

with

Denote and write

such that

A plot of for and different values is given below

